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Random Walk in Random Environment: A Counterexample without Potential

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We describe a family of random walks in random environment which have exponentially decaying correlations, nearest neighbor transition probabilities which are bounded away from 0, and are subdiffusive in any dimension $d < \infty$. The random environments have no potential in $d > 1$.

KEY WORDS: Random walk; random environment; subdiffusive; exponentially decaying correlations.

1. INTRODUCTION

Random walks in random environment have been the subject of considerable attention in recent years. Yet, few rigorous results are known about the behavior in dimensions $d > 1$. It has been shown in a momentous forthcoming article[1] that under independent environments and appropriate symmetry conditions, the mean square displacement will be asymptotically linear in time with the scaled distribution approaching that of a normal. It is believed that for models with short-range correlations, the mean square displacement also grows linearly.[2–6] In ref. 7, a family of models having spatially homogeneous random environments with exponentially decaying correlations and nearest neighbor transition probabilities which are bounded away from 0 was introduced. The random walks on these environments were shown to be subdiffusive in any dimension $d < \infty$. The environments in this family all possess potentials. The models were therefore met with some reservations as valid counterexamples.

The purpose of this article is to construct a family of models with the same features as above, but where the associated random environments do

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not possess potentials (for $d > 1$). These models are obtained by perturbing the environments in ref. 7 by independent environments so that the random walks retain their subdiffusive behavior. A major part of the construction and proof for these models resembles that in ref. 7; the reader is referred there for additional background.

The models considered in ref. 7 are a special case of random walk on a random hillside. In these systems, one starts with a random function $V: \mathbf{R}^d \to \mathbf{R}$ (the hillside or potential), defines

$$\alpha(x, y) = \exp[-\beta V((x+y)/2)] \tag{1}$$

for $x, y \in \mathbf{Z}^d$ with $|x-y| = 1$, and for convenience sets $\alpha(x, y) = 0$ otherwise. The $\alpha(x, y)$ are nonnegative, so if we let

$$\alpha(x) = \sum_y \alpha(x, y)$$

and

$$p(x, y) = \alpha(x, y)/\alpha(x),$$

then

$$p(x, y) \geqslant 0 \qquad \text{and} \qquad \sum_y p(x, y) = 1,$$

i.e., p is a transition probability. From p one constructs a random walk in random environment in the usual way: if $X(n) = x$ (that is, the particle is at x at time n), then the probability it will jump to y at time $n+1$ is $p(x, y)$ and is independent of what happened before time n. The reader should note that the definition of p is unchanged if we replace α by

$$\bar{\alpha}(x, y) = \exp\{-\beta[V((x+y)/2) - V(x)]\}, \tag{2}$$

since the extra factor will cancel when one normalizes. The value of $p(x, y)$ therefore depends only on the increments $V((x+\cdot)) - V(x)$. Assume that $X(0) = 0$.

To construct the potential V used in ref. 7, let $k(z)$, $z \in \mathbf{Z}^d$, be independent random variables with

$$\begin{aligned} &P[k(z) = 0] = 1 - \delta, \\ &P[k(z) = k] = \delta\varepsilon(1-\varepsilon)^{k-1}, \qquad k = 1, 2, \ldots. \end{aligned} \tag{3}$$

We abbreviate these probabilities by p_k. Here $0 < \delta < 1$ and $0 < \varepsilon \leqslant 1/2$. One may think of V as being the surface of a (random) moon, with $k(z)$

giving the radius of the crater centered at z. If one lets $|x| = |x_1| + \cdots + |x_d|$, then the function

$$\varphi_k(x) = \min\{|x| - k, 0\} \tag{4}$$

gives the depth of the (square) crater of radius k centered at 0. Define the surface of our moon by

$$V(x) = \min_z \varphi_{k(z)}(x - z), \tag{5}$$

where the minimum is taken over z in $\mathbf{Z}^d$.

We note that V as defined here has slope $\leqslant 1$, and so, on account of (2),

$$p(x, y) \geqslant e^{-\beta/2}/2de^{\beta/2} = (2de^{\beta})^{-1} \tag{6}$$

for $|x - y| = 1$. From the above definition it is clear that the increments in V have exponentially decaying correlations. The following result from ref. 7 shows that random walks in these environments are subdiffusive.

Theorem A. Suppose that $0 < \delta < 1$, $0 < \varepsilon \leqslant 1/2$, and $N > 0$. If $\beta \geqslant 2(N + d + 1)$, then

$$P[\max_{j \leqslant n} |X(j)| \geqslant n^{1/N}] \to 0 \qquad \text{as} \quad n \to \infty \tag{7}$$

The random walk $X(n)$ has been constructed from the random potential $V(x)$. The presence of this potential can be thought of as placing a long-range constraint on $p(x, y)$. One can therefore consider this random walk as having a "random potential" rather than a "random force." One can, however, modify this example to a random walk $Y(n)$ on $\mathbf{Z}^d$ with probabilities $p'(x, y)$ constructed in terms of $\alpha'(x, y)$ in place of $\alpha(x, y)$ as above. Set

$$\alpha'(x, y) = \exp\{-\beta[V((x, y)/2) - V(x) + W(x, y)]\} \tag{8}$$

for $|x - y| = 1$, where $W(x, y)$ are random variables (which are not necessarily independent). Then, as above, set

$$\alpha'(x) = \sum_y \alpha'(x, y) \tag{9a}$$

and

$$p'(x, y) = \alpha'(x, y)/\alpha'(x). \tag{9b}$$

Of course,

$$p'(x, y) \geq 0 \qquad \text{and} \qquad \sum_y p'(x, y) = 1.$$

Note that if $0 \leq W(x, y) \leq M$ for all x, y, then

$$p'(x, y) \geq e^{-\beta M} p(x, y) \geq (2de^{\beta(M+1)})^{-1}. \tag{10}$$

We denote by $Y(n)$ the random walk in random environment corresponding to p'. Except when specified otherwise, $Y(0) = 0$ is assumed.

We prove the following analog of Theorem A.

Theorem 1. Suppose that $0 < \delta < 1$, $0 < \varepsilon \leq 1/2$, and $N > 0$. If $\beta \geq Cd(N+1)/\varepsilon$ for appropriate C, and $0 \leq W(x, y) \leq 1/4$, then

$$P[\max_{j \leq n} |Y(j)| \geq n^{1/N}] \to 0 \qquad \text{as} \quad n \to \infty. \tag{11}$$

The process $Y(j)$ has the properties we desire. As before, $\{p'(x, y): |x - y| = 1\}$ is bounded away from zero. If W is independent of V and has exponentially decaying correlations, so does $V' = (V, W)$. Of course it is easy to choose W so that α' has no potential if $d > 1$ (e.g., $W(x, y)$ i.i.d. for $|x - y| = 1$ and $(x, y) \neq (x', y')$ will suffice).

2. DEMONSTRATION OF THEOREM 1

One can prove Theorem 1 by using an argument similar to that for Theorem A. For $X(n)$, the basic plan was motivated by the guess that the largest crater a particle falls into before leaving the ball of radius r is of order $c \log r$, where $c = -2/\log(1 - \varepsilon)$ for $d > 1$. ($X(n)$ should visit on the order of r^2 sites before leaving the ball.) The time it takes to climb out of this crater is of order $e^{\beta c \log r} = r^{\beta c}$. Inverting, one obtains (7), although one actually needs the somewhat stronger assumption $\beta \geq 2(N + d + 1)$. For $Y(n)$, with $0 \leq W(x, y) \leq 1/4$, the effect of W is compensated by choosing $\beta \geq Cd(N+1)/\varepsilon$. The particle will tend to fall into the same size craters given by V as before; increasing β increases the "pull" of a crater enough to offset W.

The proof of Theorem 1 is organized as follows: Lemma 1 and Proposition 1 will give lower bounds on the rate a particle tends to fall into a crater. They correspond to the like-labeled statements in ref. 7. Once it is in a deep crater, we wish for the particle to remain trapped there for a substantial time. Proposition 2 of ref. 7 expresses the time to climb out of a hole (perhaps consisting of many craters) in terms of the equilibrium measure $\alpha(x)$ corresponding to $p(x, y)$; the presence of the potential V

allows one to compute $\alpha(x)$. The equilibrium measure for $Y(n)$ is not, however, computable in terms of V'. (This measure will not in general be "close" to $\alpha(x)$.) So the approach employed in ref. 7 will not work here. We give a different argument in Propositions 2 and 3. Theorem 1 is then shown using Propositions 1–3.

We continue to use the notation employed in Section 1. We define φ_k, V', α', and p' as before. Denote by $\mathscr{V}'$ the σ-algebra generated by V'. As usual, Ω will denote the probability space and ω its elements.

Set $B(r) = \{x \in \mathbf{Z}^d\colon |x| < r\}$. As r increases, $B(r)$ will with high probability contain deeper and more numerous craters. A particle executing the motion $Y(n)$ should on occasion fall into such deep craters. To be more explicit, introduce a_i and b_i with

$$a_i = [\log i], \qquad b_i = a_3 + \cdots + a_i, \tag{12}$$

for $i \geqslant 3$, with $[w]$ denoting the integer part of $w \in \mathbf{Z}$. From b_i, define the sets

$$B_i = \{x \in \mathbf{Z}^d\colon |x| < b_i\} \tag{13}$$

and $A_i = B_i - B_{i-1}$. By ∂B_i, we mean those $x \in \mathbf{Z}^d$ with $\operatorname{dist}(B_i, x) = 1$. Since we are unable to say much about the motion of $Y(n)$, crude arguments regarding the placement of deep craters are required. In Proposition 1, we give a lower bound on the probability that before leaving B_i, $Y(n)$ falls at least to depth a_i in a prescribed manner. Although this probability is small, it is not too much smaller than p_{a_i}, and the event will occur with probability close to one for some B_i satisfying $B(r/4) \leqslant B_i \leqslant B(r/2)$, if r is large.

We will find it useful to define

$$A(x) = \{z\colon \varphi_{k(z)}(x - z) = V(x)\}$$

if $V(x) < 0$. We will then say that "x is influenced by $A(x)$." Note that $A(x) \neq \varnothing$, and that for $|y - x| = 1$, $V(y) = V(x) - 1$ iff $|y - z| = |x - z| - 1$ for some $z \in A(x)$. In this case, $A(y) \subset A(x)$.

Lemma 1. Fix V, h, and x_0, and suppose that x_0 is influenced by A with $\operatorname{dist}(A, x_0) \geqslant h$. For $\beta \geqslant 4 \log 6d$, $0 \leqslant W(x, y) \leqslant 1/4$, and $Y(0) = x_0$,

$$P[V(Y(j)) = V(Y(0)) - j,\ j = 1,\dots, h] \geqslant (3/4)^h. \tag{14}$$

Proof. Let $\mathscr{P}_m$ denote the set of paths $(x_0,\dots, x_m)$ (i.e., $|x_j - x_{j-1}| = 1$) with $V(x_j) = V(x_0) - j$ for $j = 1,\dots, m$. For given $(x_0,\dots, x_{m-1}) \in \mathscr{P}_{m-1}$, $m \leqslant h$, let

$$B = \{x_m\colon (x_0,\dots, x_m) \in \mathscr{P}_m\}.$$

Since x_0 is influenced by A and $\operatorname{dist}(A, x_0) \geqslant h$, B is not empty.

Note that

$$V(x_m) \geqslant V(x_{m-1}) \qquad \text{if} \quad x_m \notin B.$$

So

$$\alpha'(x_{m-1}, x_m) \leqslant 1 \qquad \text{if} \quad x_m \notin B.$$

On the other hand,

$$\alpha'(x_{m-1}, x_m) \geqslant e^{\beta/4} \qquad \text{if} \quad x_m \in B.$$

Therefore, if $\beta \geqslant 4 \log 6d$,

$$\sum_{x_m \in B} p'(x_{m-1}, x_m) \geqslant |B|\, e^{\beta/4}/(2d + |B|\, e^{\beta/4}) \geqslant 3/4.$$

Inequality (14) follows by induction. ∎

We will find it convenient to introduce two variants of $V(x)$. Let

$$\begin{aligned} V_i(x) &= \min_{z \in B_i} \varphi_{k(z)}(x - z), \\ \tilde{V}_i(x) &= V_i(x) \wedge (\operatorname{dist}(\partial B_i, x) - a_i). \end{aligned} \tag{15}$$

$V_i(x)$ measures the potential at x by ignoring the effect of craters outside B_i; $\tilde{V}_i(x)$ measures the resulting potential if one in addition includes the effect of a crater of depth a_i at a site $z \in \partial B_i$ with

$$|z - x| = \operatorname{dist}(\partial B_i, x). \tag{16}$$

Equations (15) are used in Proposition 1 in the context of σ_i (defined below). Also, for Proposition 1, let

$$T_i = \min\{n: |Y(n)| = b_i\} \tag{17}$$

and

$$\begin{aligned} \sigma_i &= T_i \wedge \min\{n: |Y(n)| > b_{i-1}, V_i(Y(n)) \neq \tilde{V}_i(Y(n))\}, \\ \tau_i &= \min\{n: V(Y(n)) \leqslant -a_i\}. \end{aligned} \tag{18}$$

(If a set is empty, assign the value ∞.) The quantity inside $\min\{\cdot\}$ in the definition of σ_i is the first time at which Y visits a site in the annulus A_i which would be influenced by a crater of depth a_i at a site $z \in \partial B_i$ (if it is not already influenced by a yet deeper crater outside B_i). Note that under fixed V', these are all stopping times. Lastly, define

$$G_i = \{\omega: \tau_i \leqslant T_i\} \tag{19}$$

and

$$\mathscr{G}_i = \sigma(G_1, ..., G_i), \qquad \mathscr{V}_i' = \sigma(V, W, \{Y(n): n \leqslant \sigma_i\}). \tag{20}$$

G_i is the event that Y has fallen deeply into a hole before leaving $\bar{B}_i = B_i \cup \partial B_i$. It is easy to check that $\mathscr{G}_{i-1} \subset \mathscr{V}_i'$.

Proposition 1. For $G_i, \mathscr{G}_{i-1}$ as defined in (19)–(20) and $\beta \geqslant 4 \log 6d$,

$$P[G_i \mid \mathscr{G}_{i-1}] \geqslant \delta\varepsilon(3(1-\varepsilon)/4)^{a_i}. \tag{21}$$

The proof of Proposition 1 is identical to that in ref. 7, and is omitted. The main idea is that since Y conditioned on $\mathscr{V}'$ is a Markov chain, one can apply the strong Markov property to Y at time σ_i. One can check that either $V(Y(\sigma_i)) \leqslant -a_i$, which implies $\omega \in G_i$, or $\sigma_i \leqslant T_i$. Choose Z_i so that $|Z_i| = b_i$ and

$$|Z_i - Y(\sigma_i)| = b_i - |Y(\sigma_i)|. \tag{22}$$

On

$$K_i = \{\omega: k(Z_i) = a_i\},$$

$X(\sigma_i)$ is influenced by Z_i or some other point not in B_i. So one can apply Lemma 1 to show that on K_i,

$$P[G_i \mid \mathscr{V}_i'] \geqslant (3/4)^{a_i}.$$

One can check that

$$P[K_i \mid \mathscr{G}_{i-1}] = \delta\varepsilon(1-\varepsilon)^{a_i - 1}.$$

Therefore,

$$P[G_i \mid \mathscr{G}_{i-1}] \geqslant E[1_{K_i} P[G_i \mid \mathscr{V}_i'] \mid \mathscr{G}_{i-1}] \geqslant \delta\varepsilon(3(1-\varepsilon)/4)^{a_i}.$$

In Proposition 1, we gave a lower bound on the probability that $Y(n)$ falls at least to depth a_i before leaving B_i. In (23) of Proposition 2, we give a lower bound on the time required for $Y(n)$ to rise from a given depth under certain regularity assumptions involving the size of nearby craters. (These assumptions ensure that the motion of a particle is locally influenced by only a single crater, which allows a simple computation of the bound.) This provides the upper bound in (24) on how far $Y(n)$ can move by a given time. After the regularity assumptions are examined in Proposition 3, Proposition 1 and (24) will be applied to demonstrate Theorem 1.

Proposition 2. Fix V so that $V(0) \leqslant -f$. Also assume that (i) no craters of depth at least h intersect $\tilde{B} = \{x: |x| < 2mh\}$ and (ii) at most m craters of depth at least $f/2$ intersect $\tilde{B}$, where $f, h, m > 0$. If $Y(0) = 0$, then

$$P[V(Y(j)) \geqslant -f/2 \text{ for some } j \leqslant n] \leqslant 4dn \exp\left\{\frac{\beta}{8}\left(2 - \frac{f}{m}\right)\right\} \tag{23}$$

and

$$P\,[\max_{j \leqslant n} |Y(j)| \geqslant 2mh] \leqslant 4dn \exp\left\{\frac{\beta}{8}\left(2 - \frac{f}{m}\right)\right\} \tag{24}$$

for all $n > 0$.

Proof. First note that on account of (i) and (ii), there are no paths connecting 0 with $\tilde{B}^c$ which remain strictly below the level $-f/2$. For such a path must remain in the above m craters until reaching $\tilde{B}^c$, whereas each such crater has diameter at most $2h$. Consequently, (24) follows from (23).

To demonstrate (23), consider the set E of those depths $g > f/2$ for which if $x \in \tilde{B}$ with $V(x) = -g$, then there is at least one neighbor y of x with $V(y) = -g - 1$. If $g \notin E$, then some x with $V(x) = -g$ is at the center of a crater. So, by (ii),

$$|g \in H^c: g > f/2| \leqslant m.$$

One can therefore choose an interval $J = (g_0 - f/2m, g_0)$ with $g_0 \in [f/2 + f/2m, f]$ so that $g \in J$ implies that $g \in E$. That is, there is an unbroken sequence of depths at least $f/2m$ long so that only perhaps the greatest depth $g_0 \in E^c$.

We can now use a standard argument involving martingles. Set

$$M(j) = \exp\{c(V(Y(j)) + g_0)\}, \tag{25}$$

where $c > 0$. For $x \in J$, there is at least one neighbor y with $V(y) = V(x) - 1$. For such y, $\alpha'(x, y) \geqslant e^{\beta/4}$, whereas for other neighbors, $\alpha'(x, y) \leqslant 1$. So for $Y(j) \in \tilde{B}$ with $V(Y(j)) \in J$,

$$\frac{E[M(j+1) \mid M(j)]}{M(j)} \leqslant \frac{e^{\beta/4} e^{-c} + 2de^{c}}{e^{\beta/4} + 2d};$$

one can check that for $c = \beta/4 - \log 2d$, this equals 1. Set

$$\tilde{M}(j) = M(j) - e^{\beta/4} j.$$

The corresponding inequality

$$E[\tilde{M}(j+1) \mid \tilde{M}(j)] \leqslant \tilde{M}(j) \tag{26}$$

holds for $Y(j)\in\tilde{B}$, $V(Y(j))\in J$. It is easy to check that for this value of c, (26) also holds for $V(Y(j))\leqslant -g_0$. So $\tilde{M}(j)$ is a supermartingale for $j\leqslant T$, the first time at which $V(Y(j))\geqslant -g_0+f/2m$.

Now, since $V(0)\leqslant -f$, $\tilde{M}(0)\leqslant 1$. So by the Optional Sampling Theorem,

$$E[\tilde{M}(n\wedge T)]\leqslant \tilde{M}(0)\leqslant 1$$

for all n. Consequently by Chebychev's inequality,

$$\begin{aligned} P[T\leqslant n] &\leqslant e^{-cf/2m}E[M(n\wedge T)] \\ &\leqslant (e^{\beta/4}n+1)e^{-cf/2m} \\ &\leqslant 4dn\exp\left\{\frac{\beta}{8}\left(2-\frac{f}{m}\right)\right\}. \quad ∎ \end{aligned}$$

In the proof of Theorem 1, we will set

$$f=\frac{1}{2}\log r, \qquad h=\frac{d+1}{\varepsilon}\log r, \qquad m=\frac{10d}{\varepsilon}. \tag{27}$$

We will therefore need to establish upper bounds on the probabilities that conditions (i) and (ii) of Proposition 2 are violated for these values. This is done in Proposition 3. We set $\tilde{B}(r)=\{x: |x|<2mh\}$. We denote by H_r the set of V for which no crater of depth at least h intersects $B(r)$ and by F_r the set of V for which there are at most m craters of depth at least $f/2$ which intersect $\tilde{B}(r)+x$ for all $x\in B(r)$. Here, f, h, and m are chosen as in (27) and $+x$ denotes translation by x.

Proposition 3. (i) $P[H_r^c]\leqslant C_1/r$ and (ii) $P[F_r^c]\leqslant C_2/r$ for appropriate C_1 and C_2 depending on ε and d.

Proof. The left side of (i) is at most

$$\begin{aligned} &\sum_{j=0}^{\infty}(2(r+[h]+j))^d\,\delta\varepsilon(1-\varepsilon)^{[h]+j-1} \\ &\quad\leqslant (8r)^d\varepsilon(1-\varepsilon)^{[h]-1}\sum_{j=0}^{\infty}(1-\varepsilon)^j+8^d\varepsilon(1-\varepsilon)^{[h]-1}\sum_{j=0}^{\infty}j^d(1-\varepsilon)^j. \end{aligned} \tag{28}$$

The first term on the right side equals

$$(8r)^d(1-\varepsilon)^{[h]-1}.$$

The second term is at most

$$8^d\varepsilon(1-\varepsilon)^{[h]-1}\sum_{j=1}^{\infty} j(j+1)\cdots(j+d+1)(1-\varepsilon)^{j-1}$$
$$= d!\,(8/\varepsilon)^d(1-\varepsilon)^{[h]-1}. \tag{29}$$

Plugging in $h=((d+1)/\varepsilon)\log r$, one can check that their sum is at most C_1/r for appropriate C_1 depending on ε and d.

The left side of (ii) can be evaluated similarly. It is at most

$$(2r)^d\sum_{j_1=0}^{\infty}\cdots\sum_{j_m=0}^{\infty}\prod_{i=1}^{m}(2(2mh+[f/2]+j_i))^d\delta\varepsilon(1-\varepsilon)^{[f/2]+j_i-1}$$
$$=(2r)^d\left(\sum_{j=0}^{\infty}(2(2mh+[f/2]+j))^d\delta\varepsilon(1-\varepsilon)^{[f/2]+j-1}\right)^m.$$

Reasoning as between (28) and (29) shows that this is

$$\leqslant (2r)^d\{((16mh)^d+d!\,(8/\varepsilon)^d)(1-\varepsilon)^{[f/2]-1}\}^m.$$

Plugging in $f=\frac{1}{2}\log r$ and $h=((d+1)/\varepsilon)\log r$, one can check that this is

$$\leqslant C'(\log r)^{dm}r^{d-\varepsilon m/5},$$

where C' depends on ε, d, and m. For $m=10d/\varepsilon$, this is clearly

$$\leqslant C_2/r$$

for appropriate C_2 depending on ε and d. ∎

We now prove Theorem 1 by using Propositions 1–3 as sketched earlier.

Theorem 1. Suppose that $0<\delta<1$, $0<\varepsilon\leqslant 1/2$, and $N>0$. If $\beta\geqslant Cd(N+1)/\varepsilon$ for appropriate C, and $0\leqslant W(x,y)\leqslant 1/4$, then

$$P[\max_{j\leqslant n}|Y(j)|\geqslant n^{1/N}]\to 0 \qquad \text{as} \quad n\to\infty. \tag{30}$$

Proof. Fix r and set

$$l=\min\{i: b_i\geqslant r/4\},$$
$$L=\max\{i: b_i<r/2\}.$$

It is easy to check that for $i\leqslant L$,

$$a_i\leqslant\log r, \tag{31}$$

and consequently

$$l \geqslant \frac{r}{4 \log r}, \qquad L-l \geqslant \frac{r}{4 \log r} - 1. \tag{32}$$

Now, repeated application of Proposition 1 shows that

$$P\left[\bigcap_{i=l}^{L} G_i^c\right] \leqslant \prod_{i=l}^{L} [1 - \delta\varepsilon(3(1-\varepsilon)/4)^{a_i}] \leqslant \exp\left[-\sum_{i=l}^{L} \delta\varepsilon(3(1-\varepsilon)/4)^{a_i}\right]. \tag{33}$$

Plug in (31) and (32), and note that $\log(3(1-\varepsilon)/4) > -1$ for $\varepsilon \leqslant 1/2$. This shows that (33) is at most

$$\exp\left[-\delta\varepsilon\left(\frac{r}{4 \log r} - 1\right) r^{\log[3(1-\varepsilon)/4]}\right] \leqslant \exp(-\delta\varepsilon r^{\eta}) \tag{34}$$

for large enough r and appropriate $\eta > 0$. Setting $G = \bigcup_{i=l}^{L} G_i$, we obtain from (33)–(34) that

$$P[G^c] \leqslant \exp(-\delta\varepsilon r^{\eta}). \tag{35}$$

We restrict our attention to $Y(n)$ on G. On this set,

$$\tau_i \leqslant T_i \qquad \text{for some} \quad l \leqslant i \leqslant L. \tag{36}$$

Denote by I the first such i, and set $Y_I = Y(\tau_I)$. From the definition of τ and a,

$$V(Y_I) \leqslant -a_I \leqslant -[\log l]. \tag{37}$$

On account of (32), for r not too small, this is

$$\leqslant -\tfrac{1}{2} \log r. \tag{38}$$

We will denote by $\mu_{V'}$ the subprobability measure on $B(r)$ induced by Y_I (restricted to G) for fixed V' and by $Z(m)$ a copy of $X(m)$ with initial distribution given by $\mu_{V'}$.

Now consider $\omega \in H_r \cap F_r$. We will plug Proposition 3 into (24) of Proposition 2 with $f = \frac{1}{2} \log r$, $h = ((d+1)/\varepsilon) \log r$, and $m = 10d/\varepsilon$, and with $Y(0) = 0$ replaced by $Z(0) = Y_I$. It is not hard to verify that the conditions in Proposition 2 are satisfied. On account of (37)–(38), $Z(0) \leqslant -f$. Note that since $Y_I \in B(r/2)$,

$$\tilde{B}(r) + Y_I \subset B(r) \tag{39}$$

for r not too small. Therefore, since $V \in H_r$, it follows that (i) of Proposition 2 holds. Also, since $V \in F_r$, (ii) of Proposition 2 holds. So, applying (24) and (39), we obtain

$$P_{\mu_{V'}}[\max_{j \leqslant n} |Z(j)| \geqslant r \mid \mathscr{V}'] \leqslant 4dn \exp\left\{\frac{\beta}{8}\left(2 - \frac{\varepsilon \log r}{20d}\right)\right\} \tag{40}$$

for $\omega \in H_r \cap F_r$.

The right side of (40) does not depend on V'. Setting $n = r^N$ with N fixed, one gets

$$\begin{aligned} &P_{\mu_{V'}}[\max_{j < r^N} |Z(j)| \geqslant r \mid \mathscr{V}'] \\ &\quad \leqslant 4de^{\beta/4} r^{N - \beta\varepsilon/160d} \leqslant C_3/r \end{aligned} \tag{41}$$

for $\beta \geqslant 160d(N+1)/\varepsilon$, where C_3 does not depend on r. Conditioned on $\mathscr{V}'$, the process Y is strong Markov. The strong Markov property therefore implies that

$$P[\max_{j \leqslant r^N} |Y(j)| \geqslant r; G \mid \mathscr{V}'] \leqslant C_3/r$$

for $\omega \in H_r \cap F_r$. Consequently,

$$P[\max_{j \leqslant r^N} |Y(j)| \geqslant r; G \mid \mathscr{V}'] \leqslant C_3/r.$$

Together with (35) and Proposition 3, this shows that

$$\begin{aligned} P[\max_{j \leqslant r^N} |Y(j)| \geqslant r] &\leqslant \exp(-\delta\varepsilon r^{\eta}) + (C_1 + C_2 + C_3)/r \\ &\to 0 \qquad \text{as} \quad r \to \infty. \end{aligned} \tag{42}$$

Inverting, one obtains

$$P[\max_{j \leqslant n} |Y(j)| \geqslant n^{1/N}] \to 0 \qquad \text{as} \quad n \to \infty. \quad \blacksquare$$

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